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THE EXPRESSION OF THE nth POWER OF A NUMBER IN TERMS OF THE nth POWERS OF OTHER NUMBERS, n BEING ANY INTEGER; AND THE DEDUCTION OF SOME INTERESTING PROPERTIES OF PRIME NUMBERS.

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[Concluded from August-September Number.]

Making $m=7$, we have

$$8^n=6^{n+1}-\tfrac{7}{2}.4^{n+1}+7.2^{n+1}=6.6^n-14.4^n+14.2^n\ldots.(37),$$

where $n=5$, 3, or 1.

When m is even and n is odd, or vice versa, (34) becomes

$$(m+1)^n=(m+1)(m-1)^n-\frac{(m+1)(m)}{2}(m-3)^n$$

$$+\frac{(m+1)\,(m)(m-1)}{6}(m-5)^5-\text{ etc., }\ldots.(38).$$

COR. V. In (21) m may be made equal to n provided that $n!\;a_1a_2\ldots.a_n$ be added to the second member.

Suppose these changes to be made and denote the result by....(39).

EXAMPLES.

Making $n=3$, in (39), we have

$$(c+a_1+a_2+a_3)^3=(c+a_1+a_2)^3+(c+a_1+a_3)^3+(c+a_2+a_3)^3$$
$$-(c+a_1)^3-(c+a_2)^3-(c+a_3)^3+c^3+1.2.3.a_1a_2a_3\ldots.(40).$$

In (39), by making $a_1=a_2=\ldots.a_n$, and transposing, we have

$$n!\,a_1{}^n=(c+na_1)^n-n[c+(n-1)a_1]^n+\frac{n(n-1)}{2}[c+(n-2)a_1]^n-\ldots.$$
$$+(-1)^nc^n\ldots.(41).$$

Making $a_1=1$, we have

$$n!=(c+n)^n-n(c+n-1)^n+\ldots.-(-1)^nn(c+1)^n+(-1)^nc^n\ldots.(42).$$

Now making $c=0$, we have

$$n!=n^n-n(n-1)^n+\frac{n(n-1)}{2}(n-2)^n-\ldots.-(-1)^nn(1)^n\ldots.(43).$$

Thus, for $n=4$, we have

$$1.2.3.4=4^4-4.3^4+6.2^4-4.1^4\ldots.(44).$$

For $n=6$ and $c=5$, (42) becomes

$$1.2.3.4.5.6=11^6-6.10^6+15.9^6-20.8^6+15.7^6-6.6^6+5^6\ldots.(45).$$

SOME INTERESTING PROPERTIES OF PRIME NUMBERS.

In the following $m+1$ is supposed to be a prime number, and m', m'', etc. represent integers.

Since each of the binomial coefficients in (25), $+1$ or -1, is divisible by $m+1$), (25) may be written

$$(c+ma_1)^n+[c+(m-1)a_1]^n+\ldots.(c+a_1)^n+c^n=m'(m+1)\ldots.(46).$$

That is, the first member, in which c and a_1 may be any integers and n any integer less than m, is exactly divisible by the prime number $m+1$.

Making $c=0$ and $a_1=1$, (46) becomes

$$(m)^n+(m-1)^n+(m-2)^n+\ldots.2^n+1=m''(m+1)\ldots.(47).$$

In (46), writing $2m$ for m, making $a_1=1$, $c=-m$, and supposing n an even number, we have

$$(m)^n+(m-1)^n+(m-2)^n+\ldots.2^n+1^n=m'(2m+1)\ldots.(48),$$

where $2m+1$ is prime, and where n is any even number less than $2m$.

Thus, 17 will exactly divide

$$8^{2x}+7^{2x}+6^{2x}+5^{2x}+4^{2x}+3^{2x}+2^{2x}+1,$$

when $x=1$, 2, 3, 4, 5, 6, or 7.

The converse of either of the preceding properties is not always true; we will now deduce some properties which belong exclusively to prime numbers.

According to Wilson's theorem, m' being an integer,

$$1+n\,!=m'(n+1)\ldots.(49)$$

only when $n+1$ is a prime number.

When $n+1$ is a prime number, (42) may be written

$$n\,!=(c+n)^n+(c+n-1)^n+(c+n-2)^n+\ldots.c^n-m'(n+1)\ldots.(50).$$

Adding 1 to both sides and we readily obtain

$$(c+n)^n+(c+n-1)^n+(c+n-2)^n+\ldots.c^n+1=m'(n+1)\ldots.(51),$$

which is true only when $n+1$ is a prime number.

Making $c=0$, and we have

$$n^n+(n-1)^n+(n-2)^n+\ldots.1+1=m'(n+1)\ldots.(52).$$

That is, S_n+1 is exactly divisible by $n+1$, when the latter is a prime number, and only when it is prime, where $S_n=1+2^n+3^n\ldots.n^n$.

In (51), by writing $2n$ for n, and $-n$ for c, and reducing, we have

$$2[n^{2n}+(n-1)^{2n}+\ldots.2^{2n}+1]+1=m'(2n+1)\ldots.(53),$$

which is also true only when $2n+1$ is a prime number.

Subtract $2n+1$ from both members of (53), and divide by $2n+1^x$, we have

$$\{[n^{2n}-1]+[n-1)^{2n}-1]+\ldots.[2^{2n}-1]\}\div(2n+1)=m''.$$

That is, if each of the quantities in the [ ] is exactly divisible by $2n+1$, then $2n+1$ is a prime number. This may be considered a generalization of Fermat's Theorem.